Quiz 3 Solutions

1. The radius of a right circular cone is increasing at a rate of 1.8 cm/sec while its height is decreasing at the rate of 2.5 cm/sec. At what rate is the volume changing when the radius is 120 cms and the height is 140 cms?

   Solution. Let r denote the radius, h denote the height, and V denote the volume of the right circular cone. Then we know that

   $$V = \frac{1}{3}\pi r^2 h.$$

   Differentiating V with respect to t using chain rule, we have

   $$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial V}{\partial h}\frac{\partial h}{\partial t},$$

   that is,

   $$\frac{dV}{dt} = \frac{2\pi r h}{3}\frac{\partial r}{\partial t} + \frac{\pi r^2}{3}\frac{\partial h}{\partial t}.$$

   We know that $\frac{\partial r}{\partial t} = 1.8$ cm/sec, $\frac{\partial h}{\partial t} = -2.5$ cm/sec, and when $r = 120$ cms and $h = 140$ cms, we have that $\frac{dV}{dt} = 8160\pi \text{ cm}^3/\text{sec}$.

2. Show that the maximum value of $f(x, y, z) = x^2y^2z^2$ on a sphere of radius r centered at origin is $(r^2/3)^3$.

   Solution. The equation of a sphere of radius r centered at origin is given by

   $$x^2 + y^2 + z^2 = r^2.$$

   Let $g(x, y, z) = x^2 + y^2 + z^2$, then by Lagrange Multipliers, there is a $\lambda \in \mathbb{R}$ such that $f(x, y, z)$ attains its extremal value at a solution point (x, y, z) of the following system of equations:

   $$\begin{aligned} f_x &= \lambda g_x, \\ f_y &= \lambda g_y, \\ f_z &= \lambda g_z, \\ x^2 + y^2 + z^2 &= r^2. \end{aligned}$$

In other words, we have to solve the system

$$\begin{aligned} x(y^2z^2 - \lambda) &= 0, \\ y(x^2z^2 - \lambda) &= 0, \\ z(x^2y^2 - \lambda) &= 0, \\ x^2 + y^2 + z^2 &= r^2. \end{aligned}$$

Since we are looking for a maximum value of a non-negative function, we have $x \neq 0$, $y \neq 0$, and $z \neq 0$ (for if any one of them is 0, then $f(x, y, z) = 0$).

Consequently, we have $x^2y^2 = y^2z^2 = z^2x^2$, giving $x^2 = y^2 = z^2$. Substituting this in $g(x, y, z) = r^2$, we have that $3x^2 = r^3$, or $x = \pm\frac{r}{\sqrt{3}}$. From this, we can deduce that $y = \pm\frac{r}{\sqrt{3}}$ and $z = \pm\frac{r}{\sqrt{3}}$. Thus, there are six possible points where the maximum is attained, and the maximum is $(r^2/3)^3$.

3. Let

$$f(x, y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

   (a) Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

   (b) Does (a) contradict the theorem on equality of mixed partials? Why or why not? [6+4]

Solution. (a) For $(x, y) \neq (0, 0)$, we have that

$$f_x(x, y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2},$$

and using the fact that $f(x, y) = -f(y, x)$ we can deduce that

$$f_y(x, y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2}.$$

We know that

$$f_{xy}(0, 0) = \frac{\partial f_x}{\partial y} = \lim_{h \to 0} \frac{f_x(0, h) - f_x(0, 0)}{h} = \frac{(-h^5 - 0)/h^4}{h} = -1.$$

In a similar fashion, we can show that $f_{yx}(0,0) = 1$, which proves (a).

(b) For $(x, y) \neq (0,0)$, we have that

$$f_{xy}(x,y) = \frac{x^6 + 9x^4y^2 - 4x^2y^4 + 4y^6}{(x^2+y^2)^3}.$$

We can see that along x-axis $f_{xy} \to 1$, and along y-axis, $f_{xy} \to 4$. This means that f_{xy} is not continuous at $(0,0)$, which violates the hypothesis of the Theorem, which assumes all second-order must be continuous. Hence, (a) does not contradict the Theorem on equality of mixed partials.